

Multistability and symmetry breaking in the two-dimensional flow around a square cylinder

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We use numerical methods to study two-dimensional flow passing a square cylinder at low Reynolds numbers, and observe period-1 and -3 vortices behind the cylinder at the same Reynolds number Re . When Re increases from a small number to a critical value $Re_c \approx 320$, the system could change from bistability, which maintains the spatial symmetry, to tristability, which breaks the spatial symmetry. Our results suggest many interesting problems for further studies. [S1063-651X(99)13711-5]

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Vortex shedding and wakes behind blunt bodies are important subjects of study in aerodynamics [1]. For a long time, people have been interested in the transition from laminar wakes to turbulent flow [2] as the Reynolds number Re increases. In three-dimensional (3D) flow around a 3D cylinder, the system has effective 2D behavior at small Re and 3D behavior at large Re ; many researchers have investigated the discontinuous transition and hysteresis between 2D parallel laminar shedding and 3D shedding with mode A characteristic [3] at $Re \approx 180-190$ for 3D flow around a circular cylinder [2]. Similar results have been found in 3D flow around some rectangular cylinders [4]. However, in the periodic vortex shedding of 3D flow around a square cylinder, the measured Strouhal number S does not show a discontinuous transition [4]. Instead, S decreases continuously when Re exceeds some threshold value at $Re \approx 130$. On physical grounds, we can argue that the decreasing of S as Re increases, i.e., $dS/dRe < 0$, might lead to some instabilities, which could result in discontinuous transitions and multistability for some rectangular cylinders [4]. In order to understand more clearly the vortex shedding processes behind a square cylinder, in this paper we use numerical methods to study 2D flow around a square cylinder, which is computationally tractable. We find very interesting results including the coexistence of period-1 and -3 vortices behind the cylinder, which can trigger bistability or tristability in different regimes of Reynolds number. When the system is bistable, the spatial symmetry is still maintained. When the system is tristable, the period-3 vortex loses spatial symmetry, but the period-1 vortex maintains the spatial symmetry. Therefore, multistability with different dynamical characteristics can be found in the 2D fluid system.

We consider a 2D uniform stream passing a square cylinder with diameter D as shown in Fig. 1. In order to make the problem computationally tractable, boundaries are placed sufficiently far away from the square cylinder so that their presence has only a tiny effect on the flow. The solution domain starts $4D$ upstream of the square cylinder and uniform flow conditions, i.e., the x direction of fluid velocity U equal to inlet mean velocity U_0 and the y direction of fluid velocity V is zero, are prescribed there. The same boundary conditions are also applied to the upper and lower bound-

aries. The outlet boundary is selected at the cross section $20D$ downstream of the square cylinder. Here Neumann-type boundary conditions are used by setting $\partial U/\partial x = 0$ and $\partial V/\partial x = 0$. The flow of an incompressible Newtonian fluid with uniform density can be described by the familiar incompressible Navier-Stokes equations. For convenience of calculation, we use dimensionless Navier-Stokes equations with length normalized by D , velocity normalized by U_0 , and time normalized by D/U_0 . Then the simulated equations with dimensionless variables and parameters can be written

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0, \quad (1)$$

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = -\frac{\partial P}{\partial x} + \frac{1}{Re} \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right), \quad (2)$$

$$\frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} = -\frac{\partial P}{\partial y} + \frac{1}{Re} \left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \right), \quad (3)$$

where P is pressure and $Re = U_0 D / \nu$ is the Reynolds number, with ν being the fluid kinetic viscosity.

It is clear to see that there is a reflective symmetry about the horizontal axis in Fig. 1. This kind of spatial symmetry shall hold the measured dimensionless quantities, i.e., the Strouhal number $S (= fD/U_0)$ and lift coefficient $C_L (= 2F_L D^{-1}/U_0^2)$, equal to some fixed values. The meaning of f and F_L are the shedding frequency of vortices and the lift force acting on a unit length of a cylinder, respectively. When the fluid system shows steady wakes, S and C_L are equal to zero. When $S \neq 0$ and C_L is periodic in time, peri-

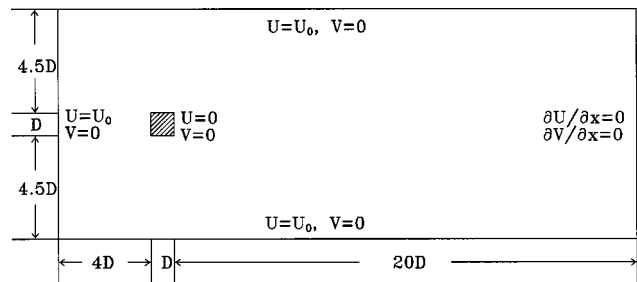


FIG. 1. Computational domain for the 2D flow around a square cylinder with diameter D .

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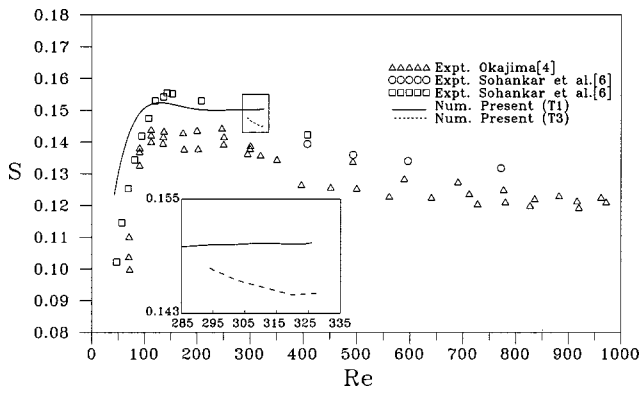


FIG. 2. The calculated and experimental data of Strouhal number S as a function of Re . The solid and dashed lines represent calculated data for a period-1 vortex ($T1$) and a period-3 vortex ($T3$), respectively. The inset is the enlarged graph in the bistable region.

odic vortex shedding occurs. The transition from the steady wake to the vortex shedding is known as Hopf bifurcation, which implies that asymmetries might appear. But if we consider the time average of C_L in a period, $\langle C_L \rangle$, we find that $\langle C_L \rangle$ is still zero immediately after Hopf bifurcation. In other words, the mirror symmetry is still maintained. Therefore, we may say that there are symmetry breaking solutions in this system if the periodic vortices with a nonzero value of $\langle C_L \rangle$.

In numerical iterations, the computational domain is first decomposed into a number of grid cells, and algebraic equa-

tions are derived in each cell to approximate Eqs. (1)–(3). The domain is divided into 250×100 uniform grid cells. To avoid the well-known checkerboard problem of pressure fields or pressure oscillation with grid length, we evaluate components of velocity at surfaces of grid cells and pressures at central points of grid cells. We thus construct a uniform staggered grid system. The difference equations used to replace Eqs. (2) and (3) are the first-order forward difference, third-order upwind difference (i.e., QUICK scheme) and second-order central difference schemes for transient terms, advective terms, and diffusive terms, respectively. The computation of incompressible fluid is more difficult than the computation of compressible fluid. To carry out the numerical computation, we employed the MAC method [5], which is a partially implicit method for solving the time-dependent equations of incompressible hydrodynamics. In order to prevent numerical instabilities, the Courant-Friedrichs-Lewy condition and the restriction on the grid Fourier numbers were imposed. According to the condition, the distance the fluid travels in one time increment must be less than one space increment, i.e., $\Delta t < \min(\Delta x/|U|, \Delta y/|V|)$. When the viscous diffusion term is more important, the necessary condition for stability is dictated by the restriction on the grid Fourier numbers, which results in $\Delta t < 0.5Re\Delta x^2\Delta y^2/(\Delta x^2 + \Delta y^2)$. The final Δt in our calculation is chosen to be the small number 0.005, which is much smaller than the stability criteria.

Numerically calculated and experimental data for Strouhal number S as a function of Re are presented in Fig. 2,

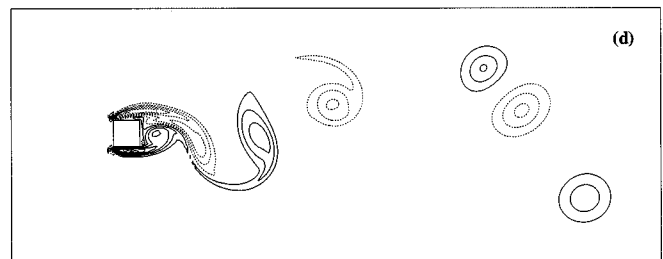
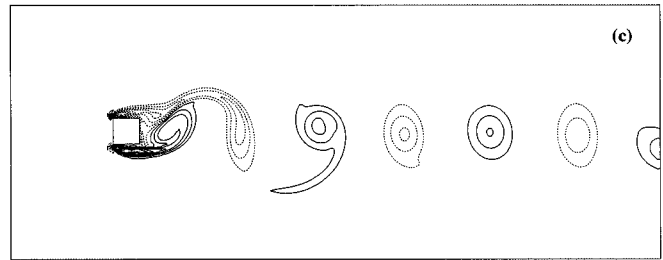
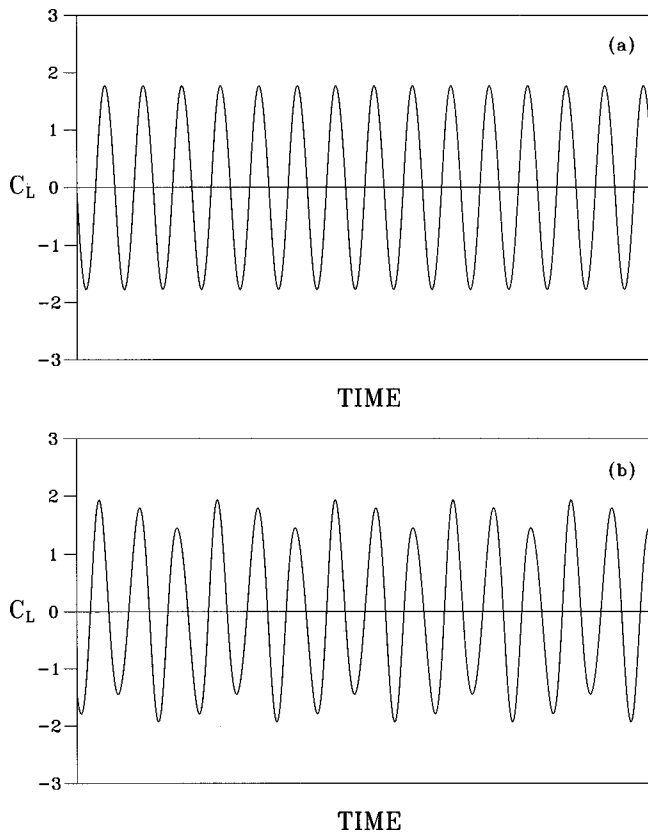


FIG. 3. (a) C_L as a function of time t for a period-1 vortex. (b) C_L as a function of t for a period-3 vortex. (c) Contour plot of vorticity for a period-1 vortex. (d) Contour plot of vorticity for a period-3 vortex. All data are obtained at $Re=300$. The dashed and solid lines in the vorticity plot represent the vortex from the top edge and bottom edge of the square cylinder, respectively.

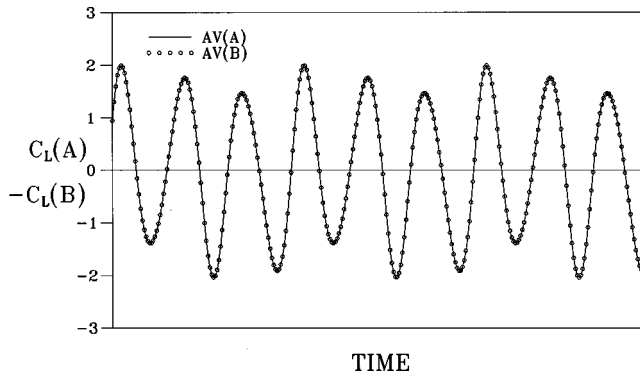


FIG. 4. Time evolution of $C_L(A)$ and $-C_L(B)$ at $Re=324$.

where the experimental data of Okajima [4] and Sohankar *et al.* [6] are for 3D flow around a square cylinder. Figure 2 shows that there is good qualitative and quantitative agreement between numerical results and experimental data in the region $Re < 250$: that is, in both cases the bifurcation from the steady wake to the laminar wake occurs at $Re \approx 42.5$ and the transition from $dS/dRe > 0$ to $dS/dRe < 0$ appears at $Re \approx 130$. When $Re \geq 250$, the qualitative behavior of simulated results is no longer similar to that of experimental data. We speculate that this is due to the difference between 2D and 3D systems, i.e., in this region the 3D flow of experimental systems cannot be considered as an effective 2D flow. This speculation is confirmed by our preliminary results for 3D simulation [7]. We find that 3D vortex shedding with mode A characteristic begins to appear at $Re \approx 250$, and the S value in the 3D simulation is lower than that in the 2D simulation. When $Re \geq 294$, vortex shedding processes for 2D flow include both periods 1 and 3, which correspond to solid and dashed lines, respectively, in Fig. 2. Period-1 and -3 vortices can coexist at the same Reynolds number and the dynamical system shows the bistability. In Fig. 3, we illustrate the coexistence of period-1 and -3 vortices via the time evolution of C_L and contour plot of vorticity at $Re = 300$. From plots of C_L vs t in Figs. 3(a) and 3(b), we can clearly distinguish period-1 and -3 patterns. In contour plots of vorticity in Figs. 3(c) and 3(d), the period-3 vortex shows larger fluctuations in the far wake than the period-1 vortex. When we evaluate time average of C_L to obtain $\langle C_L \rangle$, we find $\langle C_L \rangle = 0$ for both period-one and period-3 vortices at $Re = 300$. In other words, these two vortices sustain the spatial symmetry and they are called symmetric vortices (SV's). When Re is larger than 320, the symmetry breaking instability [8] occurs and the period-3 vortex bifurcates into two new period-3 vortex states, which break the spatial symmetry and are called asymmetric vortices (AV's); two AV states are denoted by AV(A) and AV(B). In Fig. 4, we plot $C_L(A)$ (solid line) and $-C_L(B)$ (circle) for AV(A) and AV(B), respectively, at $Re=324$ as a function of time. We find that $C_L(A) = -C_L(B)$ and $\langle C_L(A) \rangle = -\langle C_L(B) \rangle = 0.003$, which indicates symmetry breaking. The transition from symmetric to asymmetric vortices is illustrated in the $\langle C_L \rangle$ vs Re bifurcation diagram in Fig. 5. In Fig. 5, the solid line and crosses denote the period-1 SV and period-3 SV, respectively, and squares and circles denote AV(A) and AV(B), respectively. The transition from period-3 SV to period-3 AV is due to the pitchfork bifurcation. For $294 \leq Re \leq 320$, the system exhibits

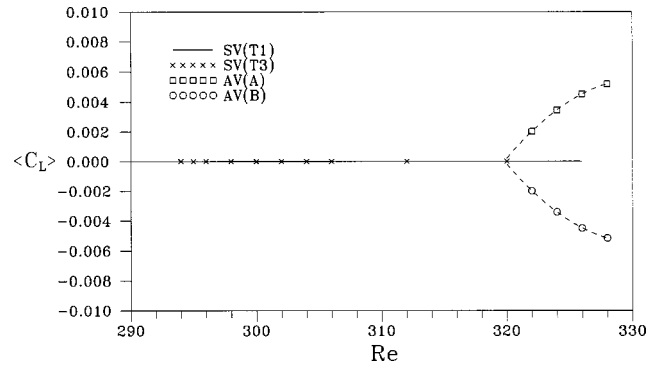


FIG. 5. $\langle C_L \rangle$ as a function of Re . The solid line and crosses (x) represent SV's with period-1 ($T1$) and period-3 ($T3$) vortices, respectively. As $Re \geq 320$, AV(A) and AV(B) with $\langle C_L \rangle \neq 0$ appear, and the system has tristability.

bistability; for $Re > 320$, the system exhibits tristability. Another way to show the transition from monostability to tristability is given in Fig. 6, in which the local maxima of C_L , $(C_L)_m$, are plotted as a function of Re . It is easy to find the correspondence between Figs. 5 and 6.

In summary, for 2D flow around a square cylinder, we use numerical methods to find steady wakes for $0 \leq Re < 42.5$, period-1 vortices for $42.5 \leq Re < 294$, bistability with only SV for $294 \leq Re \leq 320$, and tristability with both SV's and AV's for $320 < Re \leq 326$. These results are different from the well-known transition between 2D parallel vortex shedding and 3D shedding with mode A characteristic in the 3D cylinder wake. Observation of 2D AV's at low Reynolds number might imply that transition to turbulent flow in the 2D system also appears at lower Reynolds number because turbulent flow destroys spatial symmetry [9] and AV's also destroy spatial symmetry. It is worthwhile to perform further investigation on pattern competition between SV-SV, SV-AV, and AV-AV conditions by statistical analysis, i.e., spectral pattern entropy [10], which might give new insight on the pattern-forming system. It is also of interest to study 2D uniform flow around a square cylinder with low Reynolds number in experiments. Recently, Kellay *et al.* [11] studied 2D turbulence in a vertical soap film channel driven by gravity. If we can drive their system with small incoming velocity and produce laminar flow, the phenomena we have predicted might be observed in experiments.

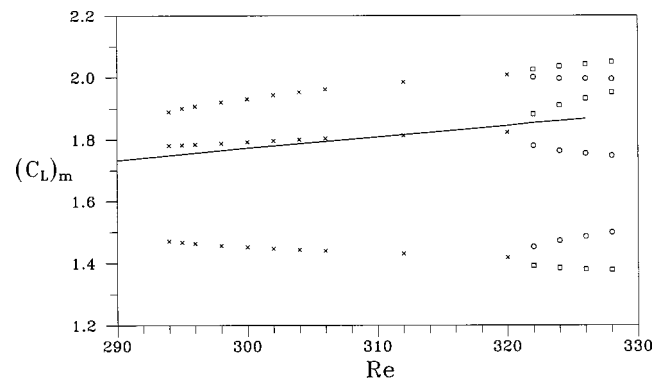


FIG. 6. The local maximum of C_L , $(C_L)_m$, as a function of Re . The solid line and crosses (x) represent SV's with period-1 ($T1$) and period-3 ($T3$) vortices, respectively. The latter bifurcates into AV(A) (square) and AV(B) (circle) a period-3 as $Re > 320$.

In conclusion, our results provide a new understanding of the 2D flow around a square cylinder, which is rather different from earlier studies for 2D flow. Our results should inspire new numerical, analytic, and experimental studies of flow around cylinders.

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